Six Challenges on the Dimension of Harmonic Measure

In $\mathbb{R}^3$: $\overline{\dim}_H\,\omega_\Omega \leq 2.99999\,9\sout{9999\,99999}$

In $\mathbb{R}^n$: $\overline{\dim}_H\,\omega_\Omega \leq n - 10^{-2n\log(n)}$

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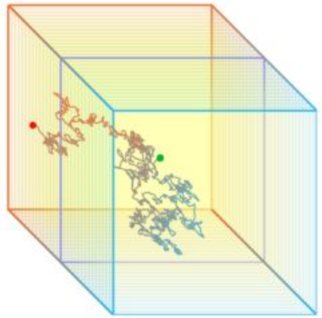
Disclaimer!

I will restrict my discussion to classical harmonic measure. There are of course many interesting relatives to investigate

elliptic measures, p-harmonic measures, caloric measure, parabolic measures, harmonic measures in higher co-dimensions, Robin harmonic measures, ...



Let $\Omega \subset \mathbb{R}^n$ be bounded, open, connected. A set $L \subset \partial\Omega$ is a **landing set for Brownian motion** if a.e. random curve drawn starting in Ω first hits the boundary of the domain in L

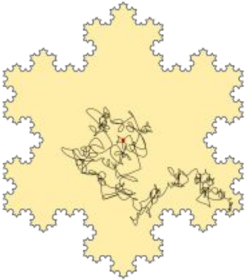
E.g. If $\Omega = (0,1)^3$ and L is the union of the 6 *open faces* of the cube (exclude the edges), then L is a landing set for Brownian motion

The **Hausdorff dimension of harmonic measure on** Ω is the smallest Hausdorff dimension of a landing set for Brownian motion in Ω

The **Hausdorff dimension of harmonic measure on** $\mathbb{R}^n$ is the largest Hausdorff dimension of harmonic measure on Ω across all domains:

$$d_n = \sup_{\Omega} \overline{\dim}_H \, \omega_\Omega, \qquad d_n \geq n-1 \text{ (e.g. take } \Omega = (0,1)^n)$$

Harmonic measure on the Koch snowflake domain



Lemma

$\dim_H \partial\Omega = \log_3(4) = 1.26185\ldots$

Theorem (Kaufman and Wu 1985)

On the snowflake domain, there is a landing set $L \subset \partial\Omega$ with $\dim_H L < \dim_H \partial\Omega$

We call this phenomena **dimension drop**

Theorem (Carleson 1985)

On the snowflake domain, there is a landing set of $L \subset \partial\Omega$ with $\dim_H L = 1$

Theorem (Makarov 1985)

On any simply connected planar domain, there is a landing set $L \subset \Omega$ with $\dim_H L = 1.$ Moreover, $\omega_\Omega(E) = 0$ for any set with $\dim_H E < 1.$

The Hausdorff dimension of harmonic measure on a simply connected domain $\overline{\dim}_H \, \omega_\Omega = 1$

Dimension of Harmonic Measure in $\mathbb{R}^n$

$$d_n = \sup_{\Omega} \overline{\dim}_H \, \omega_\Omega, \qquad d_n \geq n-1 \quad \text{(e.g. take } \Omega = (0,1)^n\text{)}$$

If $\Omega \subset \mathbb{R}^n$ and $k \geq 1$, then $\overline{\dim}_H \, \omega_{\Omega \times \mathbb{R}^k} = \overline{\dim}_H \, \omega_\Omega + k$. Hence

$$d_{n+k} \geq d_n + k \quad \text{for all } n \text{ and } k$$

Theorem (Jones and Wolff 1988)

For any $\Omega \subset \mathbb{R}^2$, there is a landing set $L \subset \partial\Omega$ with $\dim_H L \leq 1$. Thus, $d_2 = 1$.

Theorem (Wolff 1995)

There is a simply connected NTA domain Ω in $\mathbb{R}^3$ such that $\overline{\dim}_H \, \omega_\Omega > 2$. Thus, $d_n > n-1$ for all $n \geq 3$. (This doesn't say how much greater than $n-1$)

Theorem (Bourgain 1987)

For all $n \geq 3$, there is a constant $b_n > 0$ such that $d_n = n - b_n < n$.
(This doesn't say how much less than n)

What is the Hausdorff dimension of harmonic measure in $\mathbb{R}^n$?

We don't know! This question is open!

Conjecture (Bishop 1992)

The Hausdorff dimension of harmonic measure in $\mathbb{R}^3$ is 2.5.
More generally, the dimension of harmonic measure in $\mathbb{R}^n$ is $n - \frac{1}{n-1}$.

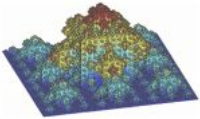
- Today I'll announce new bounds on the dimension of harmonic measure in $\mathbb{R}^n$ that are joint with Michael Albert and Alyssa Genschaw
- I'll present several challenges that could be stepping stones towards proving or disproving Bishop's conjecture
- We need more people to get involved!

Lower Bound Challenges

Numerical Experiment (Grebenkov et al. 2005)

The Hausdorff dimension of harmonic measure in $\mathbb{R}^3$ *is* ≥ 2.005

- This estimate is made by simulating Brownian motion on the fifth iteration of a cubical Koch snowflake surface.
- This bound is not yet mathematically verified.



Challenge 0

Repeat the numerical experiment using the same or different methodology to confirm or refute the 2.005 bound. Apply the method to estimate the dimension on different domains.

Challenge 1

Prove that the dimension of harmonic measure of some domain $\Omega \subset \mathbb{R}^3$ *is* $> 2 + \varepsilon$ *for some explicit number* $\varepsilon > 0$.

What is the dimension of harmonic measure in $\mathbb{R}^3$?

Theorem (Badger and Genschaw 2024)

The dimension of harmonic measure in $\mathbb{R}^3$ is $< 2.99999\,99999\,99999$

The dimension of harmonic measure in $\mathbb{R}^n$ is $< n - 10^{-n^2 \log(n)}$ provided that $n \gg 3$ ("n is much greater than 3")

- The bound in $\mathbb{R}^3$ is established by tracking through all estimates in Bourgain's proof and optimizing discrete parameters

Theorem (Albert-Badger-Genschaw 2026+)

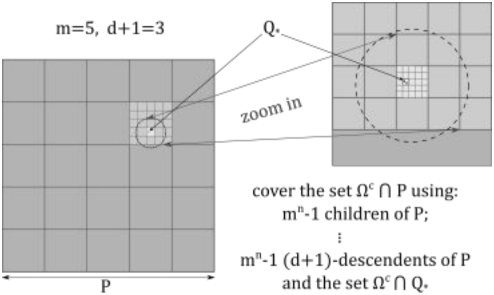
The dimension of harmonic measure in $\mathbb{R}^3$ is $< 2.99999\,9$

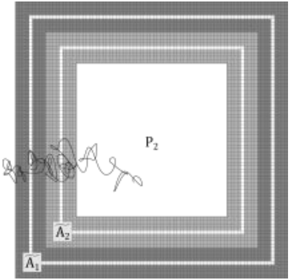
The dimension of harmonic measure in $\mathbb{R}^n$ is $< n - 10^{-2n \log(n)}$ for all (!!!) $n \geq 3$

- We employ several new strategies for estimating **non-additive** s-dimensional Hausdorff contents in $\mathbb{R}^n$ when $s < n$.

$d_3 < 2.99999\,99999\,99999, \quad d_n < n - 10^{-n^2 \log(n)}$

Idea: Introduce a model of Bourgain's alternative in which we are able to rigorously compute admissible pairs of certain constants λ and ρ for which you can show $\overline{\dim}_H \omega_\Omega \leq n - \lambda\rho/(\lambda + \rho)$.

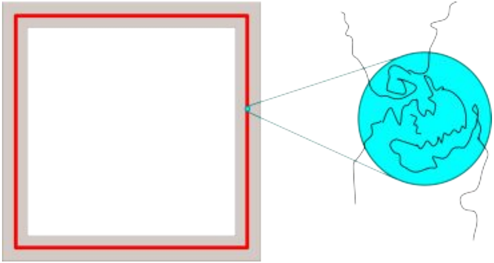




We then optimize over several discrete parameters in the the model to find the best possible bound for the dimension of harmonic measure in $\mathbb{R}^3$. In **Bourgain** and **Badger-Genschaw**, the model relies on using a fixed grid of m-adic cubes with m large depending on $n = 3$.

$d_3 < 2.99999\,9$~~9999 99999~~, $\quad d_n < n - 10^{-2n\log(n)}$

1. Redo Bourgain's analysis using **pairwise disjoint cubes** in **general position** instead of cubes in a fixed m-adic grid
2. We develop a new estimate for a **One-Ring Bourgain Alternative**: the presence of one sufficiently thin ring with lots of boundary on the outside of a cube makes it unlikely that Brownian motion will enter the inside of the cube before exiting the domain.



3. To get $d_n < n - 10^{-Cn\log(n)}$ (instead of $d_n < 10^{-Cn^2}$) it turns out that we need to use **round snapshots** instead of cubical snapshots

One-Decimal Challenge

Bishop's Conjecture: The Hausdorff dimension of harmonic measure in $\mathbb{R}^3$ is 2.5

Current Bounds: $2.005 \overset{?}{<} d_3 < 2.999999$

Challenge 2

For trivial reasons, $d_3 > 2.1$ or $d_3 < 2.9$, but we don't know the truth of either bound. Prove one of these (or both)!

Exponential Challenge

Bishop's Conjecture: The Hausdorff dimension of harmonic measure in $\mathbb{R}^n$ is $n - \frac{1}{n-1} = n - 10^{-\log_{10}(n-1)}$

Current Upper Bound: $d_n < n - 10^{-2n\log(n)}$ for all $n \geq 3$

Challenge 3

Prove that $b_n \geq 10^{-Cn}$ for all $n \geq n_0$ for some constants C and n_0 or prove any sub-exponential lower bound on b_n when n is large.

- In my opinion, it is unlikely that Challenge 3 can be resolved using a variant of Bourgain's method.

Decreasing Co-Dimension Challenge

Bourgain's constant $b_n = n - d_n$ is the amount of dimension drop in Bourgain's theorem

Bishop's Conjecture: The Hausdorff dimension of harmonic measure in $\mathbb{R}^n$ is $n - \frac{1}{n-1}$, i.e. $b_n = \frac{1}{n-1}$

Folklore Challenge: $\lim_{n\to\infty} b_n = 0$

- This challenge is intractable, because we don't have candidates for the extremal domains realizing d_n
- Here is a better challenge that we should work on first:

Challenge 4

Construct a sequence of domains Ω^n for all $n \geq 3$ such that the associated codimensions $b_{\Omega^n} := n - \overline{\dim}_H \, \omega_{\Omega^n}$ satisfy $1 > b_{\Omega^n} > b_{\Omega^{n+1}}$ for all $n \geq 3$. (I don't know how to find examples with $b_{\Omega^3} > b_{\Omega^4}$)

Cubical Snowflakes Don't Have Large Dimension

In their numerical experiment, Grebenkov et al. estimated dimension of harmonic measure on a domain G_3 that is formed by repeatedly attaching cubes to the middle third of each square face of the previous stage.

Let G_n be the analogous domain in $\mathbb{R}^n$. At each step, each face is replaced by $3^{n-1} + 2n - 1$ faces of relative size one-third:

$$\overline{\dim}_H \, \omega_{G_n} \leq \dim_H \partial G_n = \log_3(3^{n-1} + 2n - 1) \to n - 1 \quad \text{as } n \to \infty$$

- This means that the domains G_n cannot be used to solve Challenge #4
- There is no reason to believe that $\overline{\dim}_H \, G_n$ is close to d_3:

G_3 does not "expand" in every possible direction in $\mathbb{R}^3$

Packing Dimension Challenges

Packing dimension is a countably stable version of the Minkowski dimension.
For any set $\dim_H E \leq \dim_P E$

Challenge 5a

Prove or disprove: for all $n \geq 2$, there exists a constant $p_n \in (0, b_n]$ such that the packing dimension of harmonic measure on any domain is at most $n - p_n$.

- Bourgain's method cannot settle this conjecture, because of its reliance on Frostman's lemma.

Challenge 5b

Prove or disprove: the packing dimension of harmonic measure on any simply connected planar domain is 1.

- On the Koch snowflake domain, the harmonic measure lives on a set of $\mathcal{H}^1$ measure zero (McMillan's Twist Point Theorem)
- On the Koch snowflake domain, the harmonic measure vanishes on any set of σ-finite $\mathcal{P}^1$ measure (Choi 2004)

Maximal Dimension Drop Challenge

Define the **maximal dimension drop constant**:

$$m_n = \sup_{\Omega} \dim_H \partial\Omega - \overline{\dim}_H \omega_\Omega$$

There are domains with $\dim_H \partial\Omega = n$, so Bourgain's theorem implies

$$m_n \geq b_n$$

Challenge 6

Is $m_n = b_n$ or is $m_n > b_n$? In particular, looking at the case $n = 2$:

Is $m_2 = 1$ or is $m_2 > 1$? If $m_2 > 1$, then what is m_2?

- A related theorem about minimal dimension drop:
- **David-Jeznach-Julia (2023)**: for all $\varepsilon > 0$, there exists a topological Cantor set $C \subset \mathbb{R}^2$ such that the exterior domains $\Omega = \mathbb{R}^2 \setminus C$ satisfy $\overline{\dim}_H \omega_\Omega = \dim_H \partial\Omega = \dim_H C < \varepsilon$.

Questions?